場の量子論 p273check10.2

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1 本文

$$L(q,\dot{q}) = \frac{m}{2}\dot{q}^2 - \frac{m\omega^2}{2}q^2 \tag{1}$$

$$p \equiv \frac{\partial L}{\partial \dot{q}} = m\dot{q} \tag{2}$$

$$H(q,p) \equiv \dot{q}p - L(q,\dot{q}) = \frac{p^2}{2m} + \frac{m\omega^2}{2}q^2 \tag{3}$$

$$[q,p] = i, [q,q] = [p,p] = 0 \tag{4}$$

$$[H,q] = \frac{1}{2m}[p^2,q] + \frac{m\omega^2}{2}[q^2,q] \tag{5}$$

$$= \frac{1}{2m}\{ppq - pqp + pqp - qpp\} + \frac{m\omega^2}{2}\{q^3 - q^3\} \tag{6}$$

$$= \frac{1}{2m}\{p[p,q] + [p,q]p\} + 0 \tag{7}$$

$$= -\frac{i}{m}p \tag{8}$$

$$[H,p] = \frac{1}{2m}[p^2,p] + \frac{m\omega^2}{2}[q^2,p] \tag{9}$$

$$= \frac{1}{2m}\{p^3 - p^3\} + \frac{m\omega^2}{2}\{qqp - qpq + qpq - pqq\} \tag{10}$$

$$= 0 + \frac{m\omega^2}{2}\{q[q,p] + [q,p]q\} \tag{11}$$

$$= im\omega^2 q \tag{12}$$

$$a^\dagger = \sqrt{\frac{m\omega}{2}}(q - i\frac{p}{m\omega}), \quad a = \sqrt{\frac{m\omega}{2}}(q + i\frac{p}{m\omega}) \tag{13}$$

$$[H,a^\dagger] = \sqrt{\frac{m\omega}{2}}\{[H,q] - i\frac{1}{m\omega}[H,p]\} \tag{14}$$

$$= \sqrt{\frac{m\omega}{2}}\{-\frac{i}{m}p - i\frac{1}{m\omega}im\omega^2 q\} \tag{15}$$

$$= \sqrt{\frac{m\omega}{2}}\{-\omega i\frac{p}{m\omega} + \omega q\} \tag{16}$$

$$= +\omega a^\dagger \tag{17}$$

$$[H,a] = \sqrt{\frac{m\omega}{2}}\{[H,q] + i\frac{1}{m\omega}[H,p]\} \tag{18}$$
$$= \sqrt{\frac{m\omega}{2}}\{-\frac{i}{m}p + i\frac{1}{m\omega}im\omega^2 q\} \tag{19}$$
$$= \sqrt{\frac{m\omega}{2}}\{-\omega i\frac{p}{m\omega} - \omega q\} \tag{20}$$
$$= -\omega a \tag{21}$$
$$[a,a^\dagger] = \frac{m\omega}{2}[q + i\frac{p}{m\omega}, q - i\frac{p}{m\omega}] \tag{22}$$
$$= \frac{m\omega}{2}\{-i\frac{1}{m\omega}[q,p] + i\frac{1}{m\omega}[p,q]\} \tag{23}$$
$$= 1 \tag{24}$$
$$\omega(a^\dagger a + \frac{1}{2}) = \omega\{[\sqrt{\frac{m\omega}{2}}(q - i\frac{p}{m\omega})][\sqrt{\frac{m\omega}{2}}(q + i\frac{p}{m\omega})] + \frac{1}{2}\} \tag{25}$$
$$= \frac{m\omega^2}{2}(q - i\frac{p}{m\omega})(q + i\frac{p}{m\omega}) + \frac{\omega}{2} \tag{26}$$
$$= \frac{m\omega^2}{2}(q^2 - i\frac{pq}{m\omega} + i\frac{qp}{m\omega} + \frac{p^2}{m^2\omega^2}) + \frac{\omega}{2} \tag{27}$$
$$= \frac{m\omega^2}{2}\{\frac{p^2}{m^2\omega^2} + q^2 + i\frac{1}{m\omega}[q,p]\} + \frac{\omega}{2} \tag{28}$$
$$= \frac{p^2}{2m} + \frac{m\omega^2}{2}q^2 + i\frac{\omega}{2}i + \frac{\omega}{2} \tag{29}$$
$$= H \tag{30}$$
$$H|E\rangle = E|E\rangle \tag{31}$$
$$[H,a^\dagger]|E\rangle = +\omega a^\dagger|E\rangle \tag{32}$$
$$H(a^\dagger|E\rangle) - a^\dagger H|E\rangle = +\omega(a^\dagger|E\rangle) \tag{33}$$
$$H(a^\dagger|E\rangle) = (E+\omega)(a^\dagger|E\rangle) \tag{34}$$
$$[H,a]|E\rangle = -\omega a|E\rangle \tag{35}$$
$$H(a|E\rangle) - aH|E\rangle = -\omega(a|E\rangle) \tag{36}$$
$$H(a|E\rangle) = (E-\omega)(a|E\rangle) \tag{37}$$
$$a|0\rangle = 0 \tag{38}$$
$$|n\rangle = N_n(a^\dagger)^n|0\rangle \qquad (n = 0,1,2,\cdots) \tag{39}$$
$$E_n = \omega(n + \frac{1}{2}) \qquad (n = 0,1,2,\cdots) \tag{40}$$

参考文献

[1] 坂本眞人著「場の量子論-不変性と自由場を中心にして」(裳華房,2014,ISBN9784785325114)