場の量子論 p323check11.8

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1　本文

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\begin{aligned}
\phi(t,\boldsymbol{x}) &= \int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3 2E_{\boldsymbol{k}}}}\{a(\boldsymbol{k})e^{-ik_\mu x^\mu} + a^\dagger(\boldsymbol{k})e^{ik_\mu x^\mu}\} && (1)\\
&= \int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3 2E_{\boldsymbol{k}}}}\{a(\boldsymbol{k})e^{-iE_{\boldsymbol{k}}t+i\boldsymbol{k}\cdot\boldsymbol{x}} + a^\dagger(\boldsymbol{k})e^{iE_{\boldsymbol{k}}t-i\boldsymbol{k}\cdot\boldsymbol{x}}\} && (2)\\
\int d\boldsymbol{x}\frac{1}{2}(\pi(t,\boldsymbol{x}))^2 &= \int d\boldsymbol{x}\frac{1}{2}(\partial^0\phi(t,\boldsymbol{x}))^2 && (3)\\
&= \int d^3\boldsymbol{x}\frac{1}{2}\int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3 2E_{\boldsymbol{k}}}}\{a(\boldsymbol{k})(-iE_{\boldsymbol{k}})e^{-iE_{\boldsymbol{k}}t+i\boldsymbol{k}\cdot\boldsymbol{x}} + a^\dagger(\boldsymbol{k})(iE_{\boldsymbol{k}})e^{iE_{\boldsymbol{k}}t-i\boldsymbol{k}\cdot\boldsymbol{x}}\}\\
&\quad \times \int \frac{d^3\boldsymbol{k}'}{\sqrt{(2\pi)^3 2E_{\boldsymbol{k}'}}}\{a(\boldsymbol{k}')(-iE_{\boldsymbol{k}'})e^{-iE_{\boldsymbol{k}'}t+i\boldsymbol{k}'\cdot\boldsymbol{x}} + a^\dagger(\boldsymbol{k}')(iE_{\boldsymbol{k}'})e^{iE_{\boldsymbol{k}'}t-i\boldsymbol{k}'\cdot\boldsymbol{x}}\} && (4)\\
&= \frac{1}{2}\int \frac{d^3\boldsymbol{k}d^3\boldsymbol{k}'}{(2\pi)^3}\int d^3\boldsymbol{x}\frac{\sqrt{E_{\boldsymbol{k}}E_{\boldsymbol{k}'}}}{2}\{a(\boldsymbol{k})a^\dagger(\boldsymbol{k}')e^{-iE_{\boldsymbol{k}}t+iE_{\boldsymbol{k}'}t+i\boldsymbol{k}\cdot\boldsymbol{x}-i\boldsymbol{k}'\cdot\boldsymbol{x}}\\
&\quad + a^\dagger(\boldsymbol{k})a(\boldsymbol{k}')e^{iE_{\boldsymbol{k}}t-iE_{\boldsymbol{k}'}t-i\boldsymbol{k}\cdot\boldsymbol{x}+i\boldsymbol{k}'\cdot\boldsymbol{x}}\\
&\quad - a(\boldsymbol{k})a(\boldsymbol{k}')e^{-iE_{\boldsymbol{k}}t-iE_{\boldsymbol{k}'}t+i\boldsymbol{k}\cdot\boldsymbol{x}+i\boldsymbol{k}'\cdot\boldsymbol{x}}\\
&\quad - a^\dagger(\boldsymbol{k})a^\dagger(\boldsymbol{k}')e^{iE_{\boldsymbol{k}}t+iE_{\boldsymbol{k}'}t-i\boldsymbol{k}\cdot\boldsymbol{x}-i\boldsymbol{k}'\cdot\boldsymbol{x}}\} && (5)\\
&= \int d^3\boldsymbol{k}\frac{E_{\boldsymbol{k}}}{4}\{a(\boldsymbol{k})a^\dagger(\boldsymbol{k}) + a^\dagger(\boldsymbol{k})a(\boldsymbol{k}) - a(\boldsymbol{k})a(-\boldsymbol{k})e^{-2iE_{\boldsymbol{k}}t} - a^\dagger(\boldsymbol{k})a^\dagger(-\boldsymbol{k})e^{2iE_{\boldsymbol{k}}t}\} && (6)
\end{aligned}
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\begin{aligned}
\int d\boldsymbol{x}\frac{1}{2}(\nabla\phi(t,\boldsymbol{x}))^2 &= \int d^3\boldsymbol{x}\frac{1}{2}\int\frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3 2E_{\boldsymbol{k}}}}\{a(\boldsymbol{k})(i\boldsymbol{k})e^{-iE_{\boldsymbol{k}}t+i\boldsymbol{k}\cdot\boldsymbol{x}}+a^\dagger(\boldsymbol{k})(-i\boldsymbol{k})e^{iE_{\boldsymbol{k}}t-i\boldsymbol{k}\cdot\boldsymbol{x}}\} \\
&\quad\cdot\int\frac{d^3\boldsymbol{k'}}{\sqrt{(2\pi)^3 2E_{\boldsymbol{k'}}}}\{a(\boldsymbol{k'})(i\boldsymbol{k'})e^{-iE_{\boldsymbol{k'}}t+i\boldsymbol{k'}\cdot\boldsymbol{x}}+a^\dagger(\boldsymbol{k'})(-i\boldsymbol{k'})e^{iE_{\boldsymbol{k'}}t-i\boldsymbol{k'}\cdot\boldsymbol{x}}\} \qquad (7) \\
&= \frac{1}{2}\int\frac{d^3\boldsymbol{k}d^3\boldsymbol{k'}}{(2\pi)^3}\int d^3\boldsymbol{x}\frac{\boldsymbol{k}\cdot\boldsymbol{k'}}{2\sqrt{E_{\boldsymbol{k}}E_{\boldsymbol{k'}}}}\{a(\boldsymbol{k})a^\dagger(\boldsymbol{k'})e^{-iE_{\boldsymbol{k}}t+iE_{\boldsymbol{k'}}t+i\boldsymbol{k}\cdot\boldsymbol{x}-i\boldsymbol{k'}\cdot\boldsymbol{x}} \\
&\quad+a^\dagger(\boldsymbol{k})a(\boldsymbol{k'})e^{iE_{\boldsymbol{k}}t-iE_{\boldsymbol{k'}}t-i\boldsymbol{k}\cdot\boldsymbol{x}+i\boldsymbol{k'}\cdot\boldsymbol{x}} \\
&\quad-a(\boldsymbol{k})a(\boldsymbol{k'})e^{-iE_{\boldsymbol{k}}t-iE_{\boldsymbol{k'}}t+i\boldsymbol{k}\cdot\boldsymbol{x}+i\boldsymbol{k'}\cdot\boldsymbol{x}} \\
&\quad-a^\dagger(\boldsymbol{k})a^\dagger(\boldsymbol{k'})e^{iE_{\boldsymbol{k}}t+iE_{\boldsymbol{k'}}t-i\boldsymbol{k}\cdot\boldsymbol{x}-i\boldsymbol{k'}\cdot\boldsymbol{x}}\} \qquad (8) \\
&= \int d^3\boldsymbol{k}\frac{\boldsymbol{k}^2}{4E_{\boldsymbol{k}}}\{a(\boldsymbol{k})a^\dagger(\boldsymbol{k})+a^\dagger(\boldsymbol{k})a(\boldsymbol{k})+a(\boldsymbol{k})a(-\boldsymbol{k})e^{-2iE_{\boldsymbol{k}}t}+a^\dagger(\boldsymbol{k})a^\dagger(-\boldsymbol{k})e^{2iE_{\boldsymbol{k}}t}\} \qquad (9)
\end{aligned}
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\begin{aligned}
\int d\boldsymbol{x}\frac{m^2}{2}(\phi(t,\boldsymbol{x}))^2 &= \int d^3\boldsymbol{x}\frac{m^2}{2}\int\frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3 2E_{\boldsymbol{k}}}}\{a(\boldsymbol{k})e^{-iE_{\boldsymbol{k}}t+i\boldsymbol{k}\cdot\boldsymbol{x}}+a^\dagger(\boldsymbol{k})e^{iE_{\boldsymbol{k}}t-i\boldsymbol{k}\cdot\boldsymbol{x}}\} \\
&\quad\times\int\frac{d^3\boldsymbol{k'}}{\sqrt{(2\pi)^3 2E_{\boldsymbol{k'}}}}\{a(\boldsymbol{k'})e^{-iE_{\boldsymbol{k'}}t+i\boldsymbol{k'}\cdot\boldsymbol{x}}+a^\dagger(\boldsymbol{k'})e^{iE_{\boldsymbol{k'}}t-i\boldsymbol{k'}\cdot\boldsymbol{x}}\} \qquad (10) \\
&= \frac{m^2}{2}\int\frac{d^3\boldsymbol{k}d^3\boldsymbol{k'}}{(2\pi)^3}\int d^3\boldsymbol{x}\frac{1}{2\sqrt{E_{\boldsymbol{k}}E_{\boldsymbol{k'}}}}\{a(\boldsymbol{k})a^\dagger(\boldsymbol{k'})e^{-iE_{\boldsymbol{k}}t+iE_{\boldsymbol{k'}}t+i\boldsymbol{k}\cdot\boldsymbol{x}-i\boldsymbol{k'}\cdot\boldsymbol{x}} \\
&\quad+a^\dagger(\boldsymbol{k})a(\boldsymbol{k'})e^{iE_{\boldsymbol{k}}t-iE_{\boldsymbol{k'}}t-i\boldsymbol{k}\cdot\boldsymbol{x}+i\boldsymbol{k'}\cdot\boldsymbol{x}} \\
&\quad+a(\boldsymbol{k})a(\boldsymbol{k'})e^{-iE_{\boldsymbol{k}}t-iE_{\boldsymbol{k'}}t+i\boldsymbol{k}\cdot\boldsymbol{x}+i\boldsymbol{k'}\cdot\boldsymbol{x}} \\
&\quad+a^\dagger(\boldsymbol{k})a^\dagger(\boldsymbol{k'})e^{iE_{\boldsymbol{k}}t+iE_{\boldsymbol{k'}}t-i\boldsymbol{k}\cdot\boldsymbol{x}-i\boldsymbol{k'}\cdot\boldsymbol{x}}\} \qquad (11) \\
&= \int d^3\boldsymbol{k}\frac{m^2}{4E_{\boldsymbol{k}}}\{a(\boldsymbol{k})a^\dagger(\boldsymbol{k})+a^\dagger(\boldsymbol{k})a(\boldsymbol{k})+a(\boldsymbol{k})a(-\boldsymbol{k})e^{-2iE_{\boldsymbol{k}}t}+a^\dagger(\boldsymbol{k})a^\dagger(-\boldsymbol{k})e^{2iE_{\boldsymbol{k}}t}\} \qquad (12)
\end{aligned}
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\begin{aligned}
H &= \int d^3\boldsymbol{x}\{\frac{1}{2}\pi^2+\frac{1}{2}(\nabla\phi)^2+\frac{m^2}{2}\phi^2\} \qquad (13) \\
&= \int d^3\boldsymbol{k}\frac{E_{\boldsymbol{k}}}{2}\{a(\boldsymbol{k})a^\dagger(\boldsymbol{k})+a^\dagger(\boldsymbol{k})a(\boldsymbol{k})\} \qquad (14)
\end{aligned}
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\begin{aligned}
\int d\boldsymbol{x}\frac{1}{2}\pi(t,\boldsymbol{x})(\nabla\phi(t,\boldsymbol{x})) &= \int d^3\boldsymbol{x}\frac{1}{2}\int\frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3 2E_{\boldsymbol{k}}}}\{a(\boldsymbol{k})(-iE_{\boldsymbol{k}})e^{-iE_{\boldsymbol{k}}t+i\boldsymbol{k}\cdot\boldsymbol{x}}+a^\dagger(\boldsymbol{k})(iE_{\boldsymbol{k}})e^{iE_{\boldsymbol{k}}t-i\boldsymbol{k}\cdot\boldsymbol{x}}\} \\
&\quad\times\int\frac{d^3\boldsymbol{k'}}{\sqrt{(2\pi)^3 2E_{\boldsymbol{k'}}}}\{a(\boldsymbol{k'})(i\boldsymbol{k'})e^{-iE_{\boldsymbol{k'}}t+i\boldsymbol{k'}\cdot\boldsymbol{x}}+a^\dagger(\boldsymbol{k'})(-i\boldsymbol{k'})e^{iE_{\boldsymbol{k'}}t-i\boldsymbol{k'}\cdot\boldsymbol{x}}\} \qquad (15) \\
&= \frac{1}{2}\int\frac{d^3\boldsymbol{k}d^3\boldsymbol{k'}}{(2\pi)^3}\int d^3\boldsymbol{x}\frac{E_{\boldsymbol{k}}\boldsymbol{k'}}{2\sqrt{E_{\boldsymbol{k}}E_{\boldsymbol{k'}}}}\{-a(\boldsymbol{k})a^\dagger(\boldsymbol{k'})e^{-iE_{\boldsymbol{k}}t+iE_{\boldsymbol{k'}}t+i\boldsymbol{k}\cdot\boldsymbol{x}-i\boldsymbol{k'}\cdot\boldsymbol{x}} \\
&\quad -a^\dagger(\boldsymbol{k})a(\boldsymbol{k'})e^{iE_{\boldsymbol{k}}t-iE_{\boldsymbol{k'}}t-i\boldsymbol{k}\cdot\boldsymbol{x}+i\boldsymbol{k'}\cdot\boldsymbol{x}} \\
&\quad +a(\boldsymbol{k})a(\boldsymbol{k'})e^{-iE_{\boldsymbol{k}}t-iE_{\boldsymbol{k'}}t+i\boldsymbol{k}\cdot\boldsymbol{x}+i\boldsymbol{k'}\cdot\boldsymbol{x}} \\
&\quad +a^\dagger(\boldsymbol{k})a^\dagger(\boldsymbol{k'})e^{iE_{\boldsymbol{k}}t+iE_{\boldsymbol{k'}}t-i\boldsymbol{k}\cdot\boldsymbol{x}-i\boldsymbol{k'}\cdot\boldsymbol{x}}\} \qquad (16) \\
&= \int d^3\boldsymbol{k}\frac{\boldsymbol{k}}{4}\{-a(\boldsymbol{k})a^\dagger(\boldsymbol{k})-a^\dagger(\boldsymbol{k})a(\boldsymbol{k})-a(\boldsymbol{k})a(-\boldsymbol{k})e^{-2iE_{\boldsymbol{k}}t}-a^\dagger(\boldsymbol{k})a^\dagger(-\boldsymbol{k})e^{2iE_{\boldsymbol{k}}t}\} \qquad (17)
\end{aligned}
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\begin{aligned}
\int d\boldsymbol{x}\frac{1}{2}(\nabla\phi(t,\boldsymbol{x}))\pi(t,\boldsymbol{x}) &= \int d^3\boldsymbol{x}\frac{1}{2}\int\frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3 2E_{\boldsymbol{k}}}}\{a(\boldsymbol{k})(i\boldsymbol{k})e^{-iE_{\boldsymbol{k}}t+i\boldsymbol{k}\cdot\boldsymbol{x}}+a^\dagger(\boldsymbol{k})(-i\boldsymbol{k})e^{iE_{\boldsymbol{k}}t-i\boldsymbol{k}\cdot\boldsymbol{x}}\} \\
&\quad\cdot\int\frac{d^3\boldsymbol{k'}}{\sqrt{(2\pi)^3 2E_{\boldsymbol{k'}}}}\{a(\boldsymbol{k'})(-iE_{\boldsymbol{k'}})e^{-iE_{\boldsymbol{k'}}t+i\boldsymbol{k'}\cdot\boldsymbol{x}}+a^\dagger(\boldsymbol{k'})(iE_{\boldsymbol{k'}})e^{iE_{\boldsymbol{k'}}t-i\boldsymbol{k'}\cdot\boldsymbol{x}}\} \qquad (18) \\
&= \frac{1}{2}\int\frac{d^3\boldsymbol{k}d^3\boldsymbol{k'}}{(2\pi)^3}\int d^3\boldsymbol{x}\frac{E_{\boldsymbol{k}}\boldsymbol{k}}{2\sqrt{E_{\boldsymbol{k}}E_{\boldsymbol{k'}}}}\{-a(\boldsymbol{k})a^\dagger(\boldsymbol{k'})e^{-iE_{\boldsymbol{k}}t+iE_{\boldsymbol{k'}}t+i\boldsymbol{k}\cdot\boldsymbol{x}-i\boldsymbol{k'}\cdot\boldsymbol{x}} \\
&\quad -a^\dagger(\boldsymbol{k})a(\boldsymbol{k'})e^{iE_{\boldsymbol{k}}t-iE_{\boldsymbol{k'}}t-i\boldsymbol{k}\cdot\boldsymbol{x}+i\boldsymbol{k'}\cdot\boldsymbol{x}} \\
&\quad +a(\boldsymbol{k})a(\boldsymbol{k'})e^{-iE_{\boldsymbol{k}}t-iE_{\boldsymbol{k'}}t+i\boldsymbol{k}\cdot\boldsymbol{x}+i\boldsymbol{k'}\cdot\boldsymbol{x}} \\
&\quad +a^\dagger(\boldsymbol{k})a^\dagger(\boldsymbol{k'})e^{iE_{\boldsymbol{k}}t+iE_{\boldsymbol{k'}}t-i\boldsymbol{k}\cdot\boldsymbol{x}-i\boldsymbol{k'}\cdot\boldsymbol{x}}\} \qquad (19) \\
&= \int d^3\boldsymbol{k}\frac{\boldsymbol{k}}{4}\{-a(\boldsymbol{k})a^\dagger(\boldsymbol{k})-a^\dagger(\boldsymbol{k})a(\boldsymbol{k})+a(\boldsymbol{k})a(-\boldsymbol{k})e^{-2iE_{\boldsymbol{k}}t}+a^\dagger(\boldsymbol{k})a^\dagger(-\boldsymbol{k})e^{2iE_{\boldsymbol{k}}t}\} \qquad (20)
\end{aligned}
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\begin{aligned}
P &= -\int d^3\boldsymbol{x}\{\frac{1}{2}\pi(\nabla\phi)+\frac{1}{2}(\nabla\phi)\pi\} \qquad (21) \\
&= \int d^3\boldsymbol{k}\frac{\boldsymbol{k}}{2}\{a(\boldsymbol{k})a^\dagger(\boldsymbol{k})+a^\dagger(\boldsymbol{k})a(\boldsymbol{k})\} \qquad (22)
\end{aligned}
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参考文献

[1] 坂本眞人著「場の量子論-不変性と自由場を中心にして」(裳華房,2014,ISBN9784785325114)